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Diagonalization of the elliptic Macdonald–Ruijsenaars difference system of type C_2

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Abstract
We study a pair of commuting difference operators arising from the elliptic solution of the dynamical Yang–Baxter equation of type C_2. The operators act on the space of meromorphic functions on the weight space of $\mathfrak{sp}(4, \mathbb{C})$. We show that these operators can be identified with the system by van Diejen and by Komori–Hikami with special parameters. It turns out that our case can be related to the difference Lamé operator (two-body Ruijsenaars operator) and thereby we diagonalize the system on the finite-dimensional space spanned by the level-one characters of the $C_2^{(1)}$-affine Lie algebra.

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1. Introduction

The Ruijsenaars system of difference operators [1] is a difference analogue of the Calogero–Moser integrable system of differential operators. The operators of the system are defined in terms of an elliptic function, and in the trigonometric limit, they degenerate to the Macdonald q-difference operators [2]. The Ruijsenaars system has been studied extensively. In particular, Hasegawa shows that this system can be obtained as transfer matrices associated with the Sklyanin algebra [3] and Felder–Varchenko reconstructed them as transfer matrices associated with the dynamical R-matrices [4]. These two approaches are related by the vertex–IRF correspondence [5, 6].

Extending these works, in [7] we construct a pair of commuting difference operators acting on the space of functions on the C_2 type weight space. The method therein is based on the elliptic solution of the dynamical Yang–Baxter equation of type C_2 (or Boltzmann weights of the $C_2^{(1)}$ face model [8]). We have also shown that the space spanned by the level-one characters of the affine Lie algebra $\widehat{\mathfrak{sp}}(4, \mathbb{C})$ is invariant under the action of the difference operators.

On the other hand, a generalization of the Ruijsenaars system to the BC_n case is studied by van Diejen [9] and Komori–Hikami [10, 11]. In addition to the step parameter of difference operator and the modulus of elliptic functions, the family contains nine arbitrary parameters. Komori–Hikami's construction can be regarded as an elliptic generalization of a Dunkl type operator approach to Macdonald systems, which have been extensively used by Cherednik [12] (see [13] for the BC_n case).

This paper has two goals. One is to establish the relationship between our system of difference operators and the van Diejen–Komori–Hikami system. The other is to diagonalize our difference operators on the finite-dimensional space spanned by theta functions. The first goal is attained in section 2 and second in section 3.

In section 2 we review the construction of the elliptic difference system of type C_2 and give a new form of our operators. After this, we will establish an identity consisting of theta functions (lemma 1), and explain how our system can be identified with from the van Diejen–Komori–Hikami system with special choice of parameters (theorem 2). That is, our approach to the difference operators as transfer matrices, based on the knowledge of the Boltzmann weights, reproduces a special case among the family of commuting operators obtained by the Dunkl-type approach. It should be also mentioned that those two approaches to the system are not yet related, although the resulting commuting operators have the relationship as above.

In section 3, we introduce the finite-dimensional space of theta functions invariant under the action of the Weyl group and its basis after Kac–Peterson [14]. Our aim is to diagonalize our operators on this space (theorem 4). This is an elliptic analogue of the eigenvalue problem of Macdonald operators on the space of symmetric polynomials. Their eigenfunctions, called Macdonald–Koornwinder polynomials, are much investigated in q-orthogonal polynomial theory [13, 15].

2. The difference operators of type C_2

2.1. Construction of the difference operators of type C_2

Let $\mathfrak{g}$ be the Lie algebra $\mathfrak{sp}(4, \mathbb{C})$, $\mathfrak{h}$ its Cartan subalgebra and $\mathfrak{h}^*$ the dual space of $\mathfrak{h}$. We realize the root system R for $(\mathfrak{g}, \mathfrak{h})$ as $R := \{\pm(\varepsilon_1 \pm \varepsilon_2), \pm 2\varepsilon_1, \pm 2\varepsilon_2\} \subset \mathfrak{h}^*$. A normalized Killing form $(\,,\,)$ is given by

$$(\varepsilon_j, \varepsilon_k) = \tfrac{1}{2}\delta_{jk} \tag{2.1}$$

and the square length of the long roots $\pm 2\varepsilon_i$ is two. We will identify the space $\mathfrak{h}$ and its dual $\mathfrak{h}^*$ via the form $(\,,\,)$. The fundamental weights are given by $\Lambda_1 = \varepsilon_1$, $\Lambda_2 = \varepsilon_1 + \varepsilon_2$. Let $\mathcal{P}_d$ be the set of weights for the fundamental representation $L(\Lambda_d)$. We have

$$\mathcal{P}_1 = \{\pm\varepsilon_1, \pm\varepsilon_2\} \qquad \mathcal{P}_2 = \{\pm(\varepsilon_1 \pm \varepsilon_2), 0\}. \tag{2.2}$$

Let d, d' be 1 or 2. The $C_2^{(1)}$ type Boltzmann weights of type (d, d') are given as follows. Fix a complex parameter $\hbar \in \mathbb{C}$. For any pair $\lambda, \mu, \nu, \kappa \in \mathfrak{h}^*$ of weights, the Boltzmann weight

$$W_{dd'}\left(\begin{matrix} \lambda & \mu \\ \kappa & \nu \end{matrix} \middle| \, u\right)$$

is a function of the spectral parameter $u \in \mathbb{C}$ and $\lambda \in \mathfrak{h}$. They satisfy the condition

$$W_{dd'}\left(\begin{matrix} \lambda & \mu \\ \kappa & \nu \end{matrix} \middle| \, u\right) = 0 \qquad \text{unless} \quad \mu - \lambda \quad \nu - \kappa \in 2\hbar\mathcal{P}_d \quad \kappa - \lambda \quad \nu - \mu \in 2\hbar\mathcal{P}_{d'}$$

and solve the Yang–Baxter equation of the face type,

$$\sum_{\eta} W_{dd'}\left(\begin{matrix} \rho & \eta \\ \sigma & \kappa \end{matrix} \middle| \, u - v\right) W_{dd''}\left(\begin{matrix} \lambda & \mu \\ \rho & \eta \end{matrix} \middle| \, u - w\right) W_{d'd''}\left(\begin{matrix} \mu & \nu \\ \eta & \kappa \end{matrix} \middle| \, v - w\right)$$

$$= \sum_{\eta} W_{d'd''}\left(\begin{matrix} \lambda & \eta \\ \rho & \sigma \end{matrix} \middle| v-w\right) W_{dd''}\left(\begin{matrix} \eta & \nu \\ \sigma & \kappa \end{matrix} \middle| u-w\right)$$

$$\times W_{dd'}\left(\begin{matrix} \lambda & \mu \\ \eta & \nu \end{matrix} \middle| u-v\right). \tag{2.3}$$

This equation is also known as the dynamical Yang–Baxter equation. Here we give the explicit formula for W_{11} and see [7] for the other type $W_{dd'}$ ($(d, d') = (1, 2),\ (2, 1),\ (2, 2)$) which are obtained by a fusion procedure. They are expressed by the Jacobi theta function $\theta_1(u) = \theta_1(u|\tau)$ with elliptic modulus τ in the upper half plane $\mathfrak{H}_+$ (see appendix B for the definition of $\theta_1(u)$). For $p, q, r, s \in \mathcal{P}$ such that $p + q = r + s$, we will write

$$s \overset{p}{\underset{r}{\boxed{u}}} q \;=\; W_{11}\left(\begin{matrix} \lambda & \lambda + 2\hbar p \\ \lambda + 2\hbar s & \lambda + 2\hbar(p+q) \end{matrix} \middle| u\right).$$

The explicit formula for W_{11} is given as follows:

$$p \overset{p}{\underset{p}{\boxed{u}}} p \;=\; \frac{\theta_1(c-u)\,\theta_1(u+\hbar)}{\theta_1(c)\,\theta_1(\hbar)} \tag{2.4}$$

$$p \overset{p}{\underset{q}{\boxed{u}}} q \;=\; \frac{\theta_1(c-u)\,\theta_1(\lambda_{p-q}-u)}{\theta_1(c)\,\theta_1(\lambda_{p-q})} \qquad (p \neq \pm q) \tag{2.5}$$

$$p \overset{q}{\underset{q}{\boxed{u}}} p \;=\; \frac{\theta_1(c-u)\,\theta_1(u)\,\theta_1(\lambda_{p-q}+\hbar)}{\theta_1(c)\,\theta_1(\hbar)\,\theta_1(\lambda_{p-q})} \qquad (p \neq \pm q) \tag{2.6}$$

$$p \overset{q}{\underset{-p}{\boxed{u}}} -q \;=\; -\frac{\theta_1(u)\,\theta_1(\lambda_{p+q}+\hbar+c-u)}{\theta_1(c)\,\theta_1(\lambda_{p+q}+\hbar)}\,\frac{\theta_1(2\lambda_p+2\hbar)}{\theta_1(2\lambda_q)}\,\frac{\prod_{r\neq\pm p}\theta_1(\lambda_{p+r}+\hbar)}{\prod_{r\neq\pm q}\theta_1(\lambda_{q+r})} \qquad (p \neq q) \tag{2.7}$$

$$p \overset{p}{\underset{-p}{\boxed{u}}} -p \;=\; \frac{\theta_1(c-u)\,\theta_1(2\lambda_p+\hbar-u)}{\theta_1(c)\,\theta_1(2\lambda_p+\hbar)}$$

$$-\frac{\theta_1(u)\,\theta_1(2\lambda_p+\hbar+c-u)}{\theta_1(c)\,\theta_1(2\lambda_p+\hbar)}\,\frac{\theta_1(2\lambda_p+2\hbar)}{\theta_1(2\lambda_p)}\prod_{q\neq\pm p}\frac{\theta_1(\lambda_{p+q}+\hbar)}{\theta_1(\lambda_{p+q})}. \tag{2.8}$$

Here the crossing parameter c is fixed to be $c := -3\hbar$.

We define the difference operators $M_d(u)$ ($u \in \mathbb{C}, d = 1, 2$) acting on the functions on $\mathfrak{h}$ by means of the Boltzmann weights of type $(1, 2)$ and $(2, 2)$:

$$(M_d(u)f)(\lambda) := \sum_{p\in\mathcal{P}_d} W_{d2}\left(\begin{matrix} \lambda & \lambda+2\hbar p \\ \lambda & \lambda+2\hbar p \end{matrix} \middle| u\right) T^{\hbar}_{2p} f(\lambda).$$

Here the shift operator $T^{\hbar}_{2p}$ is defined as

$$T^{\hbar}_{2p} f(\lambda) := f(\lambda + 2\hbar p).$$

For $\lambda \in \mathfrak{h}^*$ and $p \in \mathcal{P}_d$ ($d = 1, 2$), we put

$$\lambda_p := (\lambda, p).$$

Note that if we denote $\lambda_i = (\lambda, \varepsilon_i)$ ($i = 1, 2$) and $f(\lambda) = f(\lambda_1, \lambda_2)$, then

$$T^{\hbar}_{\pm 2\varepsilon_1} f(\lambda_1, \lambda_2) = f(\lambda_1 \pm \hbar, \lambda_2) \qquad T^{\hbar}_{\pm 2\varepsilon_2} f(\lambda_1, \lambda_2) = f(\lambda_1, \lambda_2 \pm \hbar).$$

Theorem 1 ([7]).

(i) For each $u, v \in \mathbb{C}$, *we have* $M_d(u)M_{d'}(v) = M_{d'}(v)M_d(u)$ $(d, d' = 1, 2)$.

(ii) The explicit form of $M_d(u)$ *are as follows:*

$$M_1(u) = F(u) \sum_{p\in\mathcal{P}_1} \prod_{\substack{q\in\mathcal{P}_1\\ q\neq\pm p}} \frac{\theta_1(\lambda_{p+q} - \hbar)}{\theta_1(\lambda_{p+q})} T^{\hbar}_{2p} \tag{2.9}$$

$$M_2(u) = G(u) \left(\sum_{\substack{p=\pm\varepsilon_1\\ q=\pm\varepsilon_2}} \left(\frac{\theta_1(\lambda_{p+q} - \hbar)}{\theta_1(\lambda_{p+q} + \hbar)} T^{\hbar}_{2p} T^{\hbar}_{2q} + U(\lambda_p, \lambda_q) \right) - H(u) \right). \tag{2.10}$$

Here $U(\lambda_p, \lambda_q)$ *is given by*

$$U(\lambda_p, \lambda_q) = \frac{\theta_1(2\hbar)}{\theta_1(6\hbar)} \frac{\theta_1(2\lambda_p + 2\hbar)\,\theta_1(2\lambda_q + 2\hbar)}{\theta_1(2\lambda_p)\,\theta_1(2\lambda_q)} \frac{\theta_1(\lambda_{p+q} - 5\hbar)\,\theta_1(\lambda_{p+q} + 2\hbar)}{\theta_1(\lambda_{p+q})\,\theta_1(\lambda_{p+q} + \hbar)}$$

and $F(u), G(u), H(u)$ *are the following functions depending only on* u *and* $\hbar$*:*

$$F(u) := \frac{\theta_1(u)\,\theta_1(u + 2\hbar)^2\,\theta_1(u + 4\hbar)}{\theta_1(-3\hbar)^2\,\theta_1(\hbar)^2}$$

$$G(u) := \frac{\theta_1(u - \hbar)\,\theta_1(u)^2\,\theta_1(u + \hbar)\,\theta_1(u + 2\hbar)\,\theta_1(u + 3\hbar)^2\,\theta_1(u + 4\hbar)}{\theta_1(-3\hbar)^4\,\theta_1(\hbar)^4}$$

and

$$H(u) := \frac{\theta_1(u + 6\hbar)\,\theta_1(u - 3\hbar)\,\theta_1(2\hbar)}{\theta_1(u)\,\theta_1(u + 3\hbar)\,\theta_1(6\hbar)}.$$

The following lemma is the key for the identification with van Diejen's system as well as for the diagonalization of our difference operators. The author is grateful to van Diejen for the information.

Lemma 1. *We have*

$$\sum_{\substack{p=\pm\varepsilon_1\\ q=\pm\varepsilon_2}} U(\lambda_p, \lambda_q) - \sum_{\substack{p=\pm\varepsilon_1\\ q=\pm\varepsilon_2}} \frac{\theta_1(\lambda_{p+q} - \hbar)\,\theta_1(\lambda_{p+q} + 2\hbar)}{\theta_1(\lambda_{p+q})\,\theta_1(\lambda_{p+q} + \hbar)} = K \tag{2.11}$$

where K *is a constant given by*

$$K = \frac{\theta_1(8\hbar)\,\theta_1(\hbar)}{\theta_1(6\hbar)\,\theta_1(5\hbar)} + \frac{\theta_1(5\hbar)\,\theta_1(2\hbar)}{\theta_1(4\hbar)\,\theta_1(3\hbar)} + \frac{\theta_1(6\hbar)\,\theta_1(3\hbar)}{\theta_1(5\hbar)\,\theta_1(4\hbar)} + \frac{\theta_1(4\hbar)\,\theta_1(\hbar)}{\theta_1(3\hbar)\,\theta_1(2\hbar)}.$$

Proof. Let $f(\lambda_p)$ be the left-hand side of (2.11), regarded as a function of λ_p $(p \in I)$. It is a doubly periodic function of the periods $1, \tau$. Let us show that it is entire. The apparent poles of $f(\lambda_p)$ are located at

$$\lambda_p = \lambda_q \qquad \lambda_p = \lambda_q - \hbar (p, q \in I,\ p + q \neq 0) \qquad \lambda_p = 0 \qquad (p \in I).$$

Note that $f(\lambda_p)$ is W-invariant, then the points $\lambda_p = \lambda_q$ and $\lambda_p = 0$ are regular. Also, the residue of $f(\lambda_p)$ at $\lambda_p = -\lambda_q - \hbar$ is

$$\frac{\theta_1(2\hbar)}{\theta_1(6\hbar)} \frac{\theta_1(-2\lambda_q)\,\theta_1(2\lambda_q + 2\hbar)}{\theta_1(-2\lambda_q - 2\hbar)\,\theta_1(2\lambda_q)} \frac{\theta_1(-6\hbar)\,\theta_1(\hbar)}{\theta_1(-\hbar)} - \frac{\theta_1(-2h)\,\theta_1(\hbar)}{\theta_1(-\hbar)} = 0.$$

Now we have proved that $f(\lambda_p)$ is independent of λ_p, then we consider $g(\lambda_q) = f(-\lambda_q - 2\hbar)$ as a function of λ_q $(q \neq p \in I)$:

$$g(\lambda_q) = \frac{\theta_1(2\hbar)}{\theta_1(6\hbar)}\left(\frac{\theta_1(2\lambda_q+2\hbar)\,\theta_1(2\lambda_q-2\hbar)\,\theta_1(2\lambda_q+7\hbar)}{\theta_1(2\lambda_q+4\hbar)\,\theta_1(2\lambda_q+2\hbar)\,\theta_1(2\lambda_q+\hbar)}\right.$$
$$+\frac{\theta_1(2\lambda_q+6\hbar)\,\theta_1(2\lambda_q+2\hbar)\,\theta_1(2\lambda_q-3\hbar)}{\theta_1(2\lambda_q)\,\theta_1(2\lambda_q+2\hbar)\,\theta_1(2\lambda_q+3\hbar)}$$
$$\left.+\frac{\theta_1(2\lambda_q+6\hbar)\,\theta_1(2\lambda_q-2\hbar)\,\theta_1(-3\hbar)\,\theta_1(4\hbar)}{\theta_1(2\lambda_q+4\hbar)\,\theta_1(2\lambda_q)\,\theta_1(2\hbar)\,\theta_1(3\hbar)}\right)$$
$$-\frac{\theta_1(-2\lambda_q-3\hbar)\,\theta_1(-2\lambda_q)}{\theta_1(-2\lambda_q-2\hbar)\,\theta_1(-2\lambda_q-\hbar)}$$
$$-\frac{\theta_1(2\lambda_q+\hbar)\,\theta_1(2\lambda_q+4\hbar)}{\theta_1(2\lambda_q+2\hbar)\,\theta_1(2\lambda_q+3\hbar)} - \frac{\theta_1(\hbar)\,\theta_1(4\hbar)}{\theta_1(2\hbar)\,\theta_1(3\hbar)}.$$

By the same argument we can show that $g(\lambda_q)$ is independent of λ_q. Therefore we get K by putting $\lambda_q = \hbar$ in $g(\lambda_q)$ and the proof is complete. □

2.2. Identification with van Diejen's system

We define the difference operators $\tilde{M}_d$ to be the components of $M_d(u)$ independent of u:

$$\tilde{M}_1 = \sum_{p\in\mathcal{P}_1}\prod_{\substack{q\in\mathcal{P}_1\\ q\neq\pm p}}\frac{\theta_1(\lambda_{p+q}-\hbar)}{\theta_1(\lambda_{p+q})}T^{\hbar}_{2p} \tag{2.12}$$

$$\tilde{M}_2 = \sum_{\substack{p=\pm\varepsilon_1\\ q=\pm\varepsilon_2}}\left(\frac{\theta_1(\lambda_{p+q}-\hbar)}{\theta_1(\lambda_{p+q}+\hbar)}T^{\hbar}_{2p}T^{\hbar}_{2q} + \frac{\theta_1(\lambda_{p+q}-\hbar)\,\theta_1(\lambda_{p+q}+2\hbar)}{\theta_1(\lambda_{p+q})\,\theta_1(\lambda_{p+q}+\hbar)}\right). \tag{2.13}$$

More general commuting difference operators $\mathcal{H}_1, \mathcal{H}_2$ were obtained by van Diejen and later by Komori–Hikami in a different way. In this section we identify our operators $\tilde{M}_1, \tilde{M}_2$ as van Diejen's system of difference operators with special parameter values. The operators $\mathcal{H}_1, \mathcal{H}_2$ depend on nine complex parameters μ, μ_r, μ'_r $(r = 0, 1, 2, 3)$ satisfying the condition

$$\sum_r(\mu_r + \mu'_r) = 0 \tag{2.14}$$

and are defined by

$$\mathcal{H}_1 = \sum_{\varepsilon=\pm1} w(\varepsilon x_1)v(\varepsilon x_1 + x_2)v(\varepsilon x_1 - x_2)T^{\gamma}_{\varepsilon 1}$$
$$+\sum_{\varepsilon=\pm1} w(\varepsilon x_2)v(\varepsilon x_2 + x_1)v(\varepsilon x_2 - x_1)T^{\gamma}_{\varepsilon 2} \; + \; U_{\{1,2\},1}$$
$$\mathcal{H}_2 = \sum_{\varepsilon,\varepsilon'=\pm1} w(\varepsilon x_1)w(\varepsilon' x_2)v(\varepsilon x_1 + \varepsilon' x_2)v(\varepsilon x_1 + \varepsilon' x_2 + \gamma)T^{\gamma}_{\varepsilon 1}T^{\gamma}_{\varepsilon' 2}$$
$$+U_{\{2\},1}\sum_{\varepsilon=\pm1} w(\varepsilon x_1)v(\varepsilon x_1 + x_2)v(\varepsilon x_1 - x_2)T^{\gamma}_{\varepsilon 1}$$
$$+U_{\{1\},1}\sum_{\varepsilon=\pm1} w(\varepsilon x_2)v(\varepsilon x_2 + x_1)v(\varepsilon x_2 - x_1)T^{\gamma}_{\varepsilon 2} \; + \; U_{\{1,2\},2}.$$

Here $T^{\gamma}_{\pm i}$ $(i = 1, 2)$ stand for the shift operators

$$T^{\gamma}_{\pm 1}f(x_1, x_2) = f(x_1 \pm \gamma, x_2) \qquad T^{\gamma}_{\pm 2}f(x_1, x_2) = f(x_1, x_2 \pm \gamma)$$

and

$$v(z) := \frac{\sigma(z+\mu)}{\sigma(z)} \qquad w(z) := \prod_{0\leqslant r\leqslant 3} \frac{\sigma_r(z+\mu_r)\,\sigma_r(z+\mu_r'+\gamma/2)}{\sigma_r(z)\,\sigma_r(z+\gamma/2)} \tag{2.15}$$

where $\sigma(z) = \sigma_0(z)$ denotes the sigma function with two quasi-periods $2\omega_1,\ 2\omega_2$ and $\sigma_r(z)$ $(r = 1, 2, 3)$ associated function obtained by shift of argument over the half periods (see appendix B for more detail). The functions $U_{\{j\},1}, U_{\{1,2\},j}$ $(j = 1, 2)$ are defined as follows:

$$U_{\{j\},1} = -w(x_j) - w(-x_j) \qquad (j = 1, 2)$$

$$U_{\{1,2\},1} = \sum_{0\leqslant r\leqslant 3} c_r \prod_{j=1,2} \frac{\sigma_r(\mu-\gamma/2+x_j)\,\sigma_r(\mu-\gamma/2-x_j)}{\sigma_r(-\gamma/2+x_j)\,\sigma_r(-\gamma/2-x_j)}$$

where

$$c_r = \frac{2}{\sigma(\mu)\,\sigma(\mu-\gamma)} \prod_{0\leqslant s\leqslant 3} \sigma_s(\mu_{\pi_r(s)} - \gamma/2)\,\sigma_s(\mu'_{\pi_r(s)})$$

with π_r denoting the permutation $\pi_0 = id, \pi_1 = (01)(23), \pi_2 = (02)(13), \pi_3 = (03)(12)$.

$$U_{\{1,2\},2} = \sum_{\varepsilon,\varepsilon'\in\{1,-1\}} w(\varepsilon x_1)w(\varepsilon' x_2)v(\varepsilon x_1+\varepsilon' x_2)v(-\varepsilon x_1-\varepsilon' x_2-\gamma). \tag{2.16}$$

We mention that the Komori–Hikami system in [11] is of more complicated form and has nine arbitrary parameters, that is, they removed the condition (2.14).

In $\mathcal{H}_1, \mathcal{H}_2$, we specialize parameters μ, μ_r, μ_r' $(r = 0, 1, 2, 3)$ as $\mu = -\gamma,\ \mu_r = \mu_r' = 0$. Then $w(z) = 1$ and $U_{\{1,2\},1} = 0$. Let us denote these specialized operators by $\bar{\mathcal{H}}_1, \bar{\mathcal{H}}_2$. Because of these simplifications, we immediately obtain the following from lemma 1, giving the identification of our system $\{\tilde{M}_1,\ \tilde{M}_2\}$ and van Diejen's $\{\bar{\mathcal{H}}_1,\ \bar{\mathcal{H}}_2\}$.

Theorem 2. *For a function $f(\lambda_1, \lambda_2)$ on $\mathfrak{h}$, we set $\varphi(f)(x_1, x_2)$ by*

$$\varphi(f)(x_1,x_2) := \exp\frac{\eta_1(x_1^2+x_2^2)}{\omega_1} f\left(\frac{x_1}{2\omega_1}, \frac{x_2}{2\omega_1}\right)$$

and let $\gamma = 2\omega_1\hbar$, we have

$$\varphi\,\tilde{M}_1\,\varphi^{-1} = \mathrm{e}^{2\eta_1\gamma^2/\omega_1}\,\bar{\mathcal{H}}_1$$

$$\varphi\,\tilde{M}_2\,\varphi^{-1} = \mathrm{e}^{2\eta_1\gamma^2/\omega_1}\,\left(\bar{\mathcal{H}}_2 + 2\bar{\mathcal{H}}_1\right).$$

Proof. Use the connection between the theta function and sigma function (B.8) in appendix B and (2.11) to compare (2.10) and (2.16). □

3. Diagonalization of the system

3.1. The space of theta functions

Let Q and $Q^\vee$ be the root and coroot lattice, P and $P^\vee$ the weight and coweight lattice respectively. Under the identification $\mathfrak{h} = \mathfrak{h}^*$ via the form (,), they are given by

$$P = \sum_{j=1,2} \mathbb{Z}\varepsilon_j \qquad Q^\vee = \sum_{j=1,2} \mathbb{Z}2\varepsilon_j \tag{3.1}$$

and

$$P^\vee = Q = \mathbb{Z}2\varepsilon_1 + \mathbb{Z}2\varepsilon_2 + \mathbb{Z}(\varepsilon_1+\varepsilon_2).$$

For $\beta \in \mathfrak{h}^*$, we introduce the following operators $T_{\tau\beta}$, T_β acting on the functions on $\mathfrak{h}^*$:

$$(T_\beta f)(\lambda) := f(\lambda + \beta)$$
$$(T_{\tau\beta} f)(\lambda) := \exp\left[2\pi\mathrm{i}\left((\lambda, \beta) + \frac{(\beta, \beta)}{2}\tau\right)\right] f(\lambda + \tau\beta).$$

We define the space of theta functions (of level 1) by

$$Th_1 := \left\{f \text{ is holomorphic on } \mathfrak{h}^* | T_{\tau\alpha} f = T_\alpha f = f \ (\forall \alpha \in Q^\vee)\right\}.$$

For each $\mu \in P$ and fixed $\tau \in \mathfrak{H}_+$, we define the classical theta function $\Theta_\mu(\lambda)$ of $\lambda \in \mathfrak{h}^*$ by

$$\Theta_\mu(\lambda) := \sum_{\gamma \in \mu + Q^\vee} \exp\left[2\pi\mathrm{i}\left((\gamma, \lambda) + \frac{(\gamma, \gamma)}{2}\tau\right)\right].$$

It is known that

$$\{\Theta_\mu(\lambda) | \mu \equiv 0,\ \varepsilon_1,\ \varepsilon_2,\ \varepsilon_1 + \varepsilon_2 \bmod Q^\vee\}$$

gives a basis for Th_1 over $\mathbb{C}$ [14].

Let $W \subset GL(\mathfrak{h}^*)$ denote the Weyl group for $(\mathfrak{g}, \mathfrak{h})$, and consider the W-invariants in Th_1:

$$Th_1^W := \{f \in Th_1 | f(w\lambda) = f(\lambda)\ (\forall w \in W)\}.$$

Theorem 3 ([7]). *The operators $\tilde{M}_1$, $\tilde{M}_2$ preserves Th_1^W.*

For $\mu \in P$, we define $W_\mu := \{w \in W \mid w\mu = \mu\}$ and introduce the following symmetric sum of theta functions:

$$S_\mu(\lambda) := \frac{1}{|W_\mu|} \sum_{w \in W} \Theta_{w(\mu)}(\lambda).$$

Then

$$\{S_\mu(\lambda) \mid \mu \equiv 0,\ \Lambda_1\,(= \varepsilon_1),\ \Lambda_2\,(= \varepsilon_1 + \varepsilon_2) \bmod Q^\vee\}$$

forms a basis for Th_1^W over $\mathbb{C}$.

It is known that Th_1^W is also spanned by the level-1 characters of the affine Lie algebra $\widehat{\mathfrak{sp}}(4, \mathbb{C})$. Note that $\Theta_{-\mu}(\lambda) = \Theta_\mu(\lambda)$ and $\Theta_{\varepsilon_1+\varepsilon_2}(\lambda) = \Theta_{\varepsilon_1-\varepsilon_2}(\lambda)$. So that we have

$$S_0(\lambda) = \Theta_0(\lambda) \qquad S_{\Lambda_1}(\lambda) = 2(\Theta_{\varepsilon_1}(\lambda) + \Theta_{\varepsilon_2}(\lambda)) \qquad S_{\Lambda_2}(\lambda) = 4\Theta_{\varepsilon_1+\varepsilon_2}(\lambda).$$

3.2. Diagonalization of $\tilde{M}_d$

In this section, we diagonalize the operators $\tilde{M}_d$ on the space Th_1^W.

We set

$$f_1(\lambda) := \Theta_{\varepsilon_1}(\lambda) + \Theta_{\varepsilon_2}(\lambda)$$
$$f_2(\lambda) := \Theta_0(\lambda) + \Theta_{\varepsilon_1+\varepsilon_2}(\lambda)$$
$$f_3(\lambda) := \Theta_0(\lambda) - \Theta_{\varepsilon_1+\varepsilon_2}(\lambda).$$

They are linearly independent in the space Th_1^W.

Theorem 4. *The functions $f_i(\lambda)$ $(i = 1, 2, 3)$ are common eigenfunctions of $\tilde{M}_d$:*

$$\tilde{M}_d f_i(\lambda) = E_{d,i} f_i(\lambda) \qquad (d = 1, 2,\ i = 1, 2, 3).$$

The eigenvalues are given by

$$E_{1,i} = \left(\frac{\theta_1(2\hbar)\theta_{i+1}(0)}{\theta_1(\hbar)\theta_{i+1}(\hbar)}\right)^2$$

and $E_{2,i} = 2E_{1,i}$, where the Jacobi theta functions $\theta_i(z) = \theta_i(z|\tau)$ $(i = 2, 3, 4)$ are defined as in appendix B.

We will prove this theorem by using the following three lemmas. First, we show that the operators $\tilde{M}_d$ split into two A_1-type components.

Lemma 2. *Let us denote* $\lambda_\pm := (\lambda, \varepsilon_1 \pm \varepsilon_2)$ *and define*

$$H_\pm := \frac{\theta_1(\lambda_\pm - \hbar)}{\theta_1(\lambda_\pm)} T^\hbar_{\varepsilon_1 \pm \varepsilon_2} + \frac{\theta_1(-\lambda_\pm - \hbar)}{\theta_1(-\lambda_\pm)} T^\hbar_{-(\varepsilon_1 \pm \varepsilon_2)}.$$

Then we have

$$\tilde{M}_1 = H_+ H_- \qquad \tilde{M}_2 = H_+^2 + H_-^2. \tag{3.2}$$

Proof. To prove the first identity, we note that

$$\begin{aligned}\frac{\theta_1(\lambda_+ - \hbar)}{\theta_1(\lambda_+)} T^\hbar_{\varepsilon_1+\varepsilon_2} \frac{\theta_1(\lambda_- - \hbar)}{\theta_1(\lambda_-)} T^\hbar_{\varepsilon_1-\varepsilon_2} &= \frac{\theta_1(\lambda_+ - \hbar)}{\theta_1(\lambda_+)} \frac{\theta_1((\lambda + \hbar(\varepsilon_1 + \varepsilon_2))_- - \hbar)}{\theta_1((\lambda + \hbar(\varepsilon_1 + \varepsilon_2))_-)} T^\hbar_{\varepsilon_1+\varepsilon_2} T^\hbar_{\varepsilon_1-\varepsilon_2} \\ &= \frac{\theta_1(\lambda_+ - \hbar)}{\theta_1(\lambda_+)} \frac{\theta_1(\lambda_- - \hbar)}{\theta_1(\lambda_-)} T^\hbar_{2\varepsilon_1}.\end{aligned}$$

Here we used the identity $(\varepsilon_1 + \varepsilon_2, \varepsilon_1 - \varepsilon_2) = 0$. The second identity follows from, for instance,

$$\begin{aligned}\frac{\theta_1(\lambda_+ - \hbar)}{\theta_1(\lambda_+)} T^\hbar_{\varepsilon_1+\varepsilon_2} \frac{\theta_1(\lambda_+ - \hbar)}{\theta_1(\lambda_+)} T^\hbar_{\varepsilon_1+\varepsilon_2} &= \frac{\theta_1(\lambda_+ - \hbar)}{\theta_1(\lambda_+)} \frac{\theta_1((\lambda + \hbar(\varepsilon_1 + \varepsilon_2))_+ - \hbar)}{\theta_1((\lambda + \hbar(\varepsilon_1 + \varepsilon_2))_+)} T^\hbar_{\varepsilon_1+\varepsilon_2} T^\hbar_{\varepsilon_1+\varepsilon_2} \\ &= \frac{\theta_1(\lambda_+ - \hbar)}{\theta_1(\lambda_+)} \frac{\theta_1(\lambda_+ + \hbar - \hbar)}{\theta_1(\lambda_+ + \hbar)} T^\hbar_{2(\varepsilon_1+\varepsilon_2)} \\ &= \frac{\theta_1(\lambda_+ - \hbar)}{\theta_1(\lambda_+ + \hbar)} T^\hbar_{2\varepsilon_1} T^\hbar_{2\varepsilon_2}.\end{aligned}$$

Here we used the identity $(\varepsilon_1 + \varepsilon_2, \varepsilon_1 + \varepsilon_2) = 1$. □

Second, we consider the eigenvalue problem for the A_1-type difference operator (difference Lamé or two-body Ruijsenaars operator)

$$\frac{\theta_1(z - \ell\hbar)}{\theta_1(z)} f(z + \hbar) + \frac{\theta_1(z + \ell\hbar)}{\theta_1(z)} f(z - \hbar) = E f(z). \tag{3.3}$$

Lemma 3. *For the special coupling constant* $\ell = 1$, *the functions*

$$\theta_i(z) \qquad (i = 2, 3, 4)$$

are solutions of equation (3.3) with eigenvalues

$$E = E_i = \frac{\theta_1(2\hbar)\,\theta_i(0)}{\theta_1(\hbar)\,\theta_i(\hbar)} \qquad (i = 2, 3, 4).$$

Proof. We note that the functions $\theta_2(z)$, $\theta_3(z)$, and $\theta_4(z)$ can be rewritten in $\mathrm{e}^{cz}\theta_1(z + t)$ up to a constant, where

$$(t, c) = \left(\tfrac{1}{2}, 0\right) \quad \left(\frac{1+\tau}{2}, \pi\mathrm{i}\right) \quad \text{and} \quad \left(\frac{\tau}{2}, \pi\mathrm{i}\right) \tag{3.4}$$

respectively (see the formula (B.3) in appendix B). Let (t, c) be one of these, and we denote by $g(z)$ the function which obtained by the action of the A_1-type difference operator (3.3) with $\ell = 1$ to $\mathrm{e}^{cz}\theta_1(z + t)$:

$$g(z) := \frac{\theta_1(z - \hbar)\,\theta_1(z + t + \hbar)}{\theta_1(z)} \mathrm{e}^{c(z+\hbar)} + \frac{\theta_1(z + \hbar)\,\theta_1(z + t - \hbar)}{\theta_1(z)} \mathrm{e}^{c(z-\hbar)}.$$

This is holomorphic and doubly quasi-periodic function:

$$g(z + 1) = -\mathrm{e}^{c} g(z) \qquad g(z + \tau) = -\mathrm{e}^{\pi\mathrm{i}\tau - 2\pi\mathrm{i}(z+t) + c\tau} g(z).$$

Moreover, $g(z) = 0$ at $z = -t$. Therefore, $g(z)$ is equal to $e^{cz}\theta_1(z+t)$ up to a constant, which is the value of

$$\frac{\theta_1(z-\hbar)\,\theta_1(z+t+\hbar)}{\theta_1(z)\,\theta_1(z+t)}e^{c\hbar} + \frac{\theta_1(z+\hbar)\,\theta_1(z+t-\hbar)}{\theta_1(z)\,\theta_1(z+t)}e^{-c\hbar} \tag{3.5}$$

at any chosen point. If we choose $z = \hbar$, then the first term in (3.5) vanishes and we have

$$\frac{\theta_1(2\hbar)\,\theta_1(t)}{\theta_1(\hbar)\,\theta_1(\hbar+t)}e^{-c\hbar} = \frac{\theta_1(2\hbar)\,\theta_i(0)}{\theta_1(\hbar)\,\theta_i(\hbar)}$$

where $i = 2, 3$ and 4 corresponding to the values of (t, c) in (3.4), as an eigenvalue. □

Remark. This can be regarded as a special case of Felder–Varchenko's study [16]. They expressed the solutions of (3.3) in terms of the algebraic Bethe ansatz method, which is originally developed and applied to the spin chain model. In fact, the operator in the left-hand side of (3.3) can be regarded as the transfer matrix of the simplest spin chain, that is, it consists of only one site of freedom with spin $\ell = 1$. In this case, the Bethe ansatz equation

$$\frac{\theta_1(t-\hbar)}{\theta_1(t+\hbar)} = e^{2\hbar c} \tag{3.6}$$

is exactly the same as the condition in function (3.5) but does not have a pole at $z = -t$.

Because of lemma 3, the product of the theta functions

$$\theta_i(\lambda_+)\theta_j(\lambda_-) \qquad (i, j = 2, 3, 4)$$

are simultaneous eigenfunctions of the operators H_+^2, H_-^2 and H_+H_- with eigenvalues

$$\frac{\theta_1(2\hbar)^2\,\theta_i(0)^2}{\theta_1(\hbar)^2\,\theta_i(\hbar)^2} \qquad \frac{\theta_1(2\hbar)^2\,\theta_j(0)^2}{\theta_1(\hbar)^2\,\theta_j(\hbar)^2} \qquad \text{and} \qquad \frac{\theta_1(2\hbar)^2\,\theta_i(0)\,\theta_j(0)}{\theta_1(\hbar)^2\,\theta_i(\hbar)\,\theta_j(\hbar)}$$

respectively. Finally, we shall establish the relationship of these Bethe ansatz solutions and the bases of Th_1^W.

Lemma 4. *The functions $f_i(\lambda) \in Th_1^W$ are expressed in terms of the Jacobi theta functions as follows:*

$$f_1(\lambda) = \theta_2(\lambda_+)\theta_2(\lambda_-) \qquad f_2(\lambda) = \theta_3(\lambda_+)\theta_3(\lambda_-) \qquad f_3(\lambda) = \theta_4(\lambda_+)\theta_4(\lambda_-).$$

Proof. Because of the definitions of coroot lattice $Q^\vee$ (3.1) and Killing form (2.1), each basis of Th_1 is expressed as

$$\begin{aligned}
\Theta_0(\lambda) &= \theta_3(2\lambda_1|2\tau)\theta_3(2\lambda_2|2\tau)\\
\Theta_{\varepsilon_1}(\lambda) &= \theta_3(2\lambda_1|2\tau)\theta_2(2\lambda_2|2\tau)\\
\Theta_{\varepsilon_2}(\lambda) &= \theta_2(2\lambda_1|2\tau)\theta_3(2\lambda_2|2\tau)\\
\Theta_{\varepsilon_1+\varepsilon_2}(\lambda) &= \theta_2(2\lambda_1|2\tau)\theta_2(2\lambda_2|2\tau).
\end{aligned}$$

Here $\lambda_i = \lambda_{\varepsilon_i}$ $(i = 1, 2)$. Therefore we can prove this lemma by using the identities of theta functions (addition theorems) (B.4)–(B.7) in the appendix. □

We note that the anti-symmetric function $\Theta_{\varepsilon_1}(\lambda) - \Theta_{\varepsilon_2}(\lambda) = \theta_1(\lambda_+)\theta_1(\lambda_-)$ is also the eigenfunction with eigenvalue zero.

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Appendix A. Differential limit

Let us clarify the connection between our system of difference operators and a quantization of the Inozemtsev Hamiltonian [17, 18]. By expanding in $\hbar$ one infers that

$$\tilde{M}_1 = 4 + M_{1,2}\hbar^2 + M_{1,4}\hbar^4 + \mathrm{O}(\hbar^5)$$
$$\tilde{M}_2 = 8 + M_{2,2}\hbar^2 + M_{2,4}\hbar^4 + \mathrm{O}(\hbar^5).$$

If we abbreviate a function $f(\lambda_{\varepsilon_1 \pm \varepsilon_2})$ as $f(\pm)$, $\partial_i = \frac{\partial}{\partial \lambda_i}$ $(i = 1, 2)$, and $\theta_1'(z) = \frac{\mathrm{d}}{\mathrm{d}z}\theta_1(z)$ etc. We have

$$M_{1,2} = \partial_1^2 + \partial_2^2 - 2\left(\frac{\theta_1'}{\theta_1}(+) + \frac{\theta_1'}{\theta_1}(-)\right)\partial_1 - 2\left(\frac{\theta_1'}{\theta_1}(+) - \frac{\theta_1'}{\theta_1}(-)\right)\partial_2 + 2\left(\frac{\theta_1''}{\theta_1}(+) + \frac{\theta_1''}{\theta_1}(-)\right)$$

$$M_{2,2} = 2M_{1,2}$$

and

$$\begin{aligned}
M_{2,4} - 2M_{1,4} = {} & \partial_1^2\partial_2^2 \\
& -2\left(\frac{\theta_1'}{\theta_1}(+) - \frac{\theta_1'}{\theta_1}(-)\right)\partial_1^2\partial_2 - 2\left(\frac{\theta_1'}{\theta_1}(+) + \frac{\theta_1'}{\theta_1}(-)\right)\partial_1\partial_2^2 \\
& + \left\{2\left(\left(\frac{\theta_1'}{\theta_1}\right)^2(+) + \left(\frac{\theta_1'}{\theta_1}\right)^2(-)\right) - \left(\frac{\theta_1''}{\theta_1}(+) + 2\frac{\theta_1'}{\theta_1}(+)\frac{\theta_1'}{\theta_1}(-) + \frac{\theta_1''}{\theta_1}(-)\right)\right\}\partial_1^2 \\
& + \left\{2\left(\left(\frac{\theta_1'}{\theta_1}\right)^2(+) + \left(\frac{\theta_1'}{\theta_1}\right)^2(-)\right) - \left(\frac{\theta_1''}{\theta_1}(+) - 2\frac{\theta_1'}{\theta_1}(+)\frac{\theta_1'}{\theta_1}(-) + \frac{\theta_1''}{\theta_1}(-)\right)\right\}\partial_2^2 \\
& +4\left(\left(\frac{\theta_1'}{\theta_1}\right)^2(+) - \left(\frac{\theta_1'}{\theta_1}\right)^2(-)\right)\partial_1\partial_2 \\
& + \left\{2\left(\frac{\theta_1'\theta_1''}{\theta_1^2}(+) + \frac{\theta_1'\theta_1''}{\theta_1^2}(-)\right) + 2\left(\frac{\theta_1''}{\theta_1}(+)\frac{\theta_1'}{\theta_1}(-) + \frac{\theta_1'}{\theta_1}(+)\frac{\theta_1''}{\theta_1}(-)\right)\right. \\
& \left. \quad - 4\left(\left(\frac{\theta_1'}{\theta_1}\right)^3(+) + \left(\frac{\theta_1'}{\theta_1}\right)^3(-)\right)\right\}\partial_1 \\
& + \left\{2\left(\frac{\theta_1'\theta_1''}{\theta_1^2}(+) - \frac{\theta_1'\theta_1''}{\theta_1^2}(-)\right) - 2\left(\frac{\theta_1''}{\theta_1}(+)\frac{\theta_1'}{\theta_1}(-) - \frac{\theta_1'}{\theta_1}(+)\frac{\theta_1''}{\theta_1}(-)\right)\right. \\
& \left. \quad - 4\left(\left(\frac{\theta_1'}{\theta_1}\right)^3(+) - \left(\frac{\theta_1'}{\theta_1}\right)^3(-)\right)\right\}\partial_2 \\
& +\frac{1}{2}\left(\frac{\theta_1^{(4)}}{\theta_1}(+) + \frac{\theta_1^{(4)}}{\theta_1}(-)\right) - 4\left(\frac{\theta_1'''\theta_1'}{\theta_1^2}(+) + \frac{\theta_1'''\theta_1'}{\theta_1^2}(-)\right) \\
& +2\left(\frac{\theta_1''\theta_1'^2}{\theta_1^3}(+) + \frac{\theta_1''\theta_1'^2}{\theta_1^3}(-)\right) - 2\frac{\theta_1''}{\theta_1}(+)\frac{\theta_1''}{\theta_1}(-).
\end{aligned}$$

We set $\Delta = \theta_1(+)\theta_1(-)$, then

$$\begin{aligned}
\Delta^{-1}\cdot M_{2,2}\cdot\Delta &= \partial_1^2 + \partial_2^2 + 4\left(\left(\frac{\theta_1''}{\theta_1} - \frac{\theta_1'^2}{\theta_1^2}\right)(+) + \left(\frac{\theta_1''}{\theta_1} - \frac{\theta_1'^2}{\theta_1^2}\right)(-)\right) \\
&= \partial_1^2 + \partial_2^2 + 4\left((\log\theta_1)''(+) + (\log\theta_1)''(-)\right)
\end{aligned} \qquad \text{(A.1)}$$

$$
\begin{aligned}
\Delta^{-1}\cdot(M_{2,4}-2M_{1,4})\cdot\Delta &= \partial_1^2\partial_2^2\\
&\quad +4\left(\left(\frac{\theta_1''}{\theta_1}+\frac{\theta_1'^2}{\theta_1^2}\right)(+)-\left(\frac{\theta_1''}{\theta_1}+\frac{\theta_1'^2}{\theta_1^2}\right)(-)\right)\partial_1\partial_2\\
&\quad +2\left(\left(\frac{\theta_1'''}{\theta_1}-3\frac{\theta_1''\theta_1'}{\theta_1^2}+2\frac{\theta_1'^3}{\theta_1^3}\right)(+)+\left(\frac{\theta_1'''}{\theta_1}-3\frac{\theta_1''\theta_1'}{\theta_1^2}+2\frac{\theta_1'^3}{\theta_1^3}\right)(-)\right)\partial_1\\
&\quad +2\left(\left(\frac{\theta_1'''}{\theta_1}-3\frac{\theta_1''\theta_1'}{\theta_1^2}+2\frac{\theta_1'^3}{\theta_1^3}\right)(+)-\left(\frac{\theta_1'''}{\theta_1}-3\frac{\theta_1''\theta_1'}{\theta_1^2}+2\frac{\theta_1'^3}{\theta_1^3}\right)(-)\right)\partial_2\\
&\quad +2\left(\frac{\theta_1^{(4)}}{\theta_1}(+)+\frac{\theta_1^{(4)}}{\theta_1}(-)\right)-8\left(\frac{\theta_1'''\theta_1'}{\theta_1^2}(+)+\frac{\theta_1'''\theta_1'}{\theta_1^2}(-)\right)\\
&\quad -2\left(\left(\frac{\theta_1''}{\theta_1}\right)^2(+)+\left(\frac{\theta_1''}{\theta_1}\right)^2(-)\right)-8\frac{\theta_1''}{\theta_1}(+)\frac{\theta_1''}{\theta_1}(-)\\
&\quad +16\left(\frac{\theta_1''\theta_1'^2}{\theta_1^3}(+)+\frac{\theta_1''\theta_1'^2}{\theta_1^3}(-)\right)+8\left(\frac{\theta_1'^2}{\theta_1^2}(+)\frac{\theta_1''}{\theta_1}(-)+\frac{\theta_1''}{\theta_1}(+)\frac{\theta_1'^2}{\theta_1^2}(-)\right)\\
&\quad -8\left(\left(\frac{\theta_1'}{\theta_1}\right)^4(+)+\left(\frac{\theta_1'}{\theta_1}\right)^4(-)\right)+8\left(\frac{\theta_1'}{\theta_1}\right)^2(+)\left(\frac{\theta_1'}{\theta_1}\right)^2(-)\\
&= \partial_1^2\partial_2^2\\
&\quad +4\{(\log\theta_1)''(+)-(\log\theta_1)''(-)\}\partial_1\partial_2\\
&\quad +2\left\{(\log\theta_1)'''(+)+(\log\theta_1)'''(-)\right\}\partial_1+2\{(\log\theta_1)'''(+)-(\log\theta_1)'''(-)\}\partial_2\\
&\quad +2\{(\log\theta_1)^{(4)}(+)+(\log\theta_1)^{(4)}(-)\}\\
&\quad +4\{(\log\theta_1)''(+)-(\log\theta_1)''(-)\}^2\\
&= \left\{\partial_1\partial_2+2\left((\log\theta_1)''(+)-(\log\theta_1)''(-)\right)\right\}^2.
\end{aligned}
$$

The complete integrable Hamiltonian of type BC_n is introduced by Olshanetsky–Perelomov [19], and later generated by Inozemtsev–Meshcheryakov [17, 18]. In the rank two case, the Hamiltonian is

$$
\begin{aligned}
H &= -\tfrac{1}{2}(\partial_1^2+\partial_2^2)+g(g-1)\left(\wp(x_1+x_2)+\wp(x_1-x_2)\right)\\
&\quad +\sum_{0\leqslant r\leqslant 3} g_r(g_r-1)(\wp(\omega_r+x_1)+\wp(\omega_r+x_2))
\end{aligned}
$$

where $\wp(x)$ denotes the Weierstrass $\wp$-function with two periods $2\omega_1$ and $2\omega_2$, and $\omega_0 = 0$, $\omega_3 = -\omega_1-\omega_2$. By the connection between theta function and $\wp$ function (B.9) in appendix B, our differential limit (A.1) is identified with this Hamiltonian for the special coupling constants $g(g-1) = 2$, and $g_r(g_r-1) = 0$ $(0 \leqslant r \leqslant 3)$.

Appendix B. Theta function

We establish notations and identities on the theta functions [20]. The Jacobi theta functions are defined for $\tau \in \mathfrak{H}_+$ as follows:

$$
\begin{aligned}
\theta_1(z|\tau) &= \sum_{k\in\mathbb{Z}} \exp[2\pi\mathrm{i}((z+\tfrac{1}{2})(k+\tfrac{1}{2})+\tfrac{1}{2}(k+\tfrac{1}{2})^2\tau)]\\
\theta_2(z|\tau) &= \sum_{k\in\mathbb{Z}} \exp[2\pi\mathrm{i}(z(k+\tfrac{1}{2})+\tfrac{1}{2}(k+\tfrac{1}{2})^2\tau)]
\end{aligned}
$$

$$\theta_3(z|\tau) = \sum_{k\in\mathbb{Z}} \exp\left[2\pi \mathrm{i}\left(zk + \frac{k^2}{2}\tau\right)\right]$$
$$\theta_4(z|\tau) = \sum_{k\in\mathbb{Z}} \exp\left[2\pi \mathrm{i}\left(\left(z+\frac{1}{2}\right)k + \frac{k^2}{2}\tau\right)\right].$$

Note that $\theta_1(z)$ is odd and the other three are even. These functions has quasi-periodicity:

$$\theta_1(z+m|\tau) = (-1)^m \theta_1(z|\tau) \tag{B.1}$$
$$\theta_1(z+m\tau|\tau) = (-1)^m \mathrm{e}^{-\pi \mathrm{i} m^2\tau - 2\pi \mathrm{i} m z}\theta_1(z|\tau) \tag{B.2}$$

($m \in \mathbb{Z}$), while other three can be expressed by $\theta_1(z)$

$$\begin{aligned} &\theta_1(z+\tfrac{1}{2}|\tau) = \theta_2(z|\tau) \\ &\theta_1\left(z+\frac{\tau}{2}\middle|\tau\right) = \mathrm{i}\mathrm{e}^{-\pi\mathrm{i}(z+\frac{\tau}{4})}\theta_4(z|\tau) \\ &\theta_1\left(z+\frac{1}{2}+\frac{\tau}{2}\middle|\tau\right) = \mathrm{e}^{-\pi\mathrm{i}(z+\frac{\tau}{4})}\theta_3(z|\tau). \end{aligned} \tag{B.3}$$

We use these identities in the computations in lemma 4:

$$\theta_4(x|\tau)\theta_4(y|\tau) = \theta_3(x+y|2\tau)\theta_3(x-y|2\tau) - \theta_2(x+y|2\tau)\theta_2(x-y|2\tau) \tag{B.4}$$
$$\theta_3(x|\tau)\theta_3(y|\tau) = \theta_3(x+y|2\tau)\theta_3(x-y|2\tau) + \theta_2(x+y|2\tau)\theta_2(x-y|2\tau) \tag{B.5}$$
$$\theta_2(x|\tau)\theta_2(y|\tau) = \theta_3(x+y|2\tau)\theta_2(x-y|2\tau) + \theta_2(x+y|2\tau)\theta_3(x-y|2\tau) \tag{B.6}$$
$$\theta_1(x|\tau)\theta_1(y|\tau) = \theta_3(x+y|2\tau)\theta_2(x-y|2\tau) - \theta_2(x+y|2\tau)\theta_3(x-y|2\tau). \tag{B.7}$$

The sigma function $\sigma(z)$ is an entire, odd, and quasi-periodic function with two primitive quasi-periods $2\omega_1, 2\omega_2$.

$$\sigma(z+2n\omega_1+2m\omega_2) = (-1)^{n+m+nm}\mathrm{e}^{(2n\eta_1+2m\eta_2)(z+n\omega_1+m\omega_2)}\sigma(z)$$

with $\eta_i = \zeta(\omega_i)$ $(i = 1, 2)$, where $\zeta(z) = \sigma'(z)/\sigma(z)$ denotes the Weierstrass ζ-function. The connection between the Jacobi theta functions and the sigma functions are

$$\sigma(z) = \left(\exp\frac{\eta_1 z^2}{2\omega_1}\right)\frac{\theta_1(z/2\omega_1)}{\theta_1'(0)}$$
$$\sigma_r(z) = \left(\exp\frac{\eta_1 z^2}{2\omega_1}\right)\frac{\theta_{r+1}(z/2\omega_1)}{\theta_{r+1}(0)} \qquad (r = 1, 2, 3).$$

Then, for the function $v(z)$ in van Diejen's system (2.15), we have

$$v(z) := \frac{\sigma(z+\mu)}{\sigma(z)} = \left(\exp\frac{\eta_1(2z\mu+\mu^2)}{2\omega_1}\right)\frac{\theta_1((z+\mu)/2\omega_1)}{\theta_1(z/2\omega_1)}. \tag{B.8}$$

The connection with the $\wp$ function is

$$\wp(z) = -\frac{\mathrm{d}^2}{\mathrm{d}z^2}\log\sigma(z) = -\frac{1}{4\omega_1^2}\left(\frac{\mathrm{d}^2}{\mathrm{d}z^2}\log\theta_1(z/2\omega_1)\right) - \frac{\eta_1}{\omega_1}. \tag{B.9}$$

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